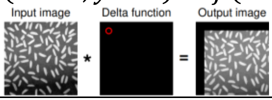
Standard calculus stuff

$e^{jz} = \cos z + j \sin z$	$\sin z = \frac{1}{2j}\left(e^{jz} - e^{-jz}\right)$	$\cos z = \frac{1}{2}\left(e^{jz} + e^{-jz}\right)$	$\mathbf{A}^{-1} = \frac{1}{\lvert\mathbf{A}\rvert}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
$\iint_{\mathcal{D}} f(x,y)\,dx\,dy = \int_{\mathcal{D}_1} f_1(x)\,dx \int_{\mathcal{D}_2} f_2(y)\,dy$		**Polar coordinates:** $x = u\cos v\,;\, y = u \sin v$	
$\iint_{\mathcal{D}} f(x,y)\,dx\,dy = \iint_{\mathcal{D}'} f\big(x(u,v), y(u,v)\big)\,J(u,v)\,du\,dv\,;\; J(u,v) = \left\lvert \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right\rvert$			

Dirac delta stuff

$\int \delta(x+a) f(x)\,dx = f(-a)$	$\delta(x+a) * \delta(x+b) = \delta(x+a+b)$
$\delta(x+u, y+v) = \delta(x+u)\delta(y+v)$	$f(x,y) = \iint f(u,v)\,\delta(x-u, y-v)\,du\,dv$
$\delta(ax) = \frac{1}{\lvert a\rvert}\delta(x);\; \delta(ax, by) = \frac{1}{\lvert ab\rvert}\delta(x,y)$	$f(x,y)\delta(x+a, y+b) = f(-a,-b)\delta(x+a, y+b)$
$\int_{\mathcal{D}} \delta(x+a)\,dx = \begin{cases} 1, & -a \in \mathcal{D} \\ 0, & -a \notin \mathcal{D} \end{cases}$	$f(x,y) * \delta(x+a, y+b) = f(x+a, y+b)$ 
$\delta(u,v) = \left\lvert\lvert\mathbf{A}^{-1}\rvert\right\rvert \delta(x,y);\; \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}$	

Convolution stuff

Convolution (linear & shift-invariant)**:** $f * g(x,y) = \iint f(u,v)\,g(x-u, y-v)\,du\,dv$

Cross-correlation: $f \star g(x,y) = f(x,y) * g(-x,-y)^* = \iint f(u,v)\,g(u-x, v-y)^*\,du\,dv$

Continuous-space Fourier transform stuff

$$F(\omega_x, \omega_y) = \iint f(x,y) e^{-jx\omega_x} e^{-jy\omega_y}\,dx\,dy \qquad f(x,y) = \frac{1}{4\pi^2}\iint F(\omega_x,\omega_y) e^{jx\omega_x} e^{jy\omega_y}\,d\omega_x\,d\omega_y$$

$$F(\omega_x,\omega_y) G(\omega_x,\omega_y) = \iint f * g(x,y) e^{-jx\omega_x} e^{-jy\omega_y}\,dx\,dy$$

- **Duality**: $F(x,y) \xrightarrow{\text{Fourier}} 4\pi^2 f(-\omega_x, -\omega_y)$

Translation: $f(x+x_0, y+y_0) \xrightarrow{\mathcal{F}} F(\omega_x,\omega_y) e^{j(x_0\omega_x + y_0\omega_y)}$

- **Modulation**: $f(x,y) e^{jx\omega_{0x} + jy\omega_{0y}} \xrightarrow{\text{Fourier}} F(\omega_x - \omega_{0x}, \omega_y - \omega_{0y})$
- **Differentiation**: $\partial_x^k \partial_y^l f(x,y) \xrightarrow{\text{Fourier}} (j\omega_x)^k (j\omega_y)^l F(\omega_x,\omega_y)$
- **Moments**: $x^k y^l f(x,y) \xrightarrow{\text{Fourier}} j^{k+l} \partial_{\omega_x}^k \partial_{\omega_y}^l F(\omega_x,\omega_y)$
- **Affine transformation**[a]: $f(\mathbf{Ar}) \xrightarrow{\text{Fourier}} \lvert\det(\mathbf{A}^{-1})\rvert\, F\left((\mathbf{A}^{-1})^{\intercal}\boldsymbol{\omega}\right)$
- **Hermitian symmetry**: $f^*(x,y) \xrightarrow{\text{Fourier}} F^*(-\omega_x, -\omega_y)$
- **Parseval theorem**: $4\pi^2 \langle f, g\rangle = \langle F, G\rangle$

[a] e.g., scaling $\Leftrightarrow$ $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$: $f(ax, by) \xrightarrow{\text{Fourier}} \lvert ab\rvert^{-1} F(a^{-1}\omega_x, b^{-1}\omega_y)$.

Affine: set $f(x,y) = g(ax, by); \mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ ...

$$\delta(x,y) \xrightarrow{\text{Fourier}} 1$$

$$1 \xrightarrow{\text{Fourier}} 4\pi^2 \delta(\omega_x, \omega_y)$$

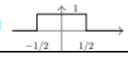
$$e^{-\frac{1}{\sigma}(x^2+y^2)} \xrightarrow{\mathcal{F}} \pi\sigma e^{-\frac{\sigma}{4}(\omega_x^2 + \omega_y^2)}$$

$$f_1(x) f_2(y) \xrightarrow{\text{Fourier}} F_1(\omega_x) F_2(\omega_y)$$

$$\text{rect}(x)\,\text{rect}(y) \xrightarrow{\text{Fourier}} \text{sinc}\left(\tfrac{\omega_x}{2\pi}\right) \text{sinc}\left(\tfrac{\omega_y}{2\pi}\right)$$

$$\text{sinc}(x)\,\text{sinc}(y) \xrightarrow{\text{Fourier}} \text{rect}\left(\tfrac{\omega_x}{2\pi}\right) \text{rect}\left(\tfrac{\omega_y}{2\pi}\right)$$

$$\textstyle\sum_{k,l} \delta(x-k, y-l) \xrightarrow{\text{Fourier}} 4\pi^2 \sum_{k,l} \delta(\omega_x - 2\pi k, \omega_y - 2\pi l)$$

$\text{rect}(x)$ is the function  $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

$\sin ax \xrightarrow{\mathcal{F}} -j\pi\big(\delta(\omega - a) - \delta(\omega + a)\big)$	$\cos ax \xrightarrow{\mathcal{F}} \pi\big(\delta(\omega - a) + \delta(\omega + a)\big)$

$$\sin(ax + by) \xrightarrow{\mathcal{F}} 2j\pi^2 \Big(\delta(\omega_x + a)\delta(\omega_y + b) - \delta(\omega_x - a)\delta(\omega_y - b)\Big)$$

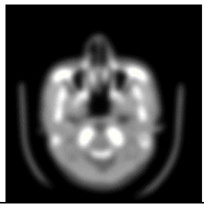
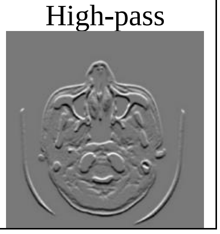
$$\cos(ax + by) \xrightarrow{\mathcal{F}} 2\pi^2 \Big(\delta(\omega_x - a)\delta(\omega_y - b) + \delta(\omega_x + a)\delta(\omega_y + b)\Big)$$

Property	Function	Fourier Transform
Inverse	$\hat{f}(t)$	$2\pi\, f(-\omega)$
Convolution	$f_1 * f_2(t)$	$\hat{f}_1(\omega)\,\hat{f}_2(\omega)$
Multiplication	$f_1(t)\,f_2(t)$	$\frac{1}{2\pi}\hat{f}_1 * \hat{f}_2(\omega)$

Property	Function	Fourier Transform
Scaling	$f(kt)$	$\left\lvert\frac{1}{k}\right\rvert \hat{f}\left(\frac{1}{k}\omega\right)$
Time derivatives	$f^{(p)}(t)$	$(i\omega)^p\,\hat{f}(\omega)$
Frequency derivatives	$(-it)^p\,f(t)$	$\hat{f}^{(p)}(\omega)$

$F(\omega_x,\omega_y) = F(-\omega_x,-\omega_y)^*$ for real-valued $f(x,y)$	$\delta(x+y) \xrightarrow{\mathcal{F}} 2\pi\delta(\omega_x - \omega_y)$

Filter stuff

Normalised gaussian filter: $h = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$	Original	Gaussian	Directional blur	High-pass
Low-pass: $H(\pm\infty, \pm\infty) = 0$ **High-pass:** $H(0,0) = 0$				

Sampling stuff

$\sin \omega x, \cos \omega x \Rightarrow$ bandlimited in $[\pm\omega]$	$\operatorname{sinc} \omega x \Rightarrow$ bandlimited in $[\pm\pi\omega]$

Shannon-Whittaker-Kotel'nikov sampling theorem

If the image $f(x, y)$ to reconstruct is such that its Fourier transform satisfies the *bandlimitation* condition

$$F(\omega_x, \omega_y) \neq 0 \quad \text{only when} \quad \begin{cases} |\omega_x| \leq \omega_x^{\max} \\ \text{and} \\ |\omega_y| \leq \omega_y^{\max} \end{cases} \quad (\text{"Nyquist zone"})$$

then the continuous image can be reconstructed *exactly from its samples alone* provided that $T_x \leq \frac{\pi}{\omega_x^{\max}}$ and $T_y \leq \frac{\pi}{\omega_y^{\max}}$.

The reconstruction formula takes the form of an infinite sum of shifted cardinal sine functions (Shannon's summation formula)

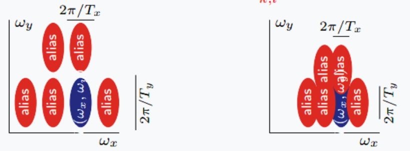
$$f(x, y) = \sum_{k,l} f(kT_x, lT_y) \operatorname{sinc}\left(\frac{x}{T_x} - k\right) \operatorname{sinc}\left(\frac{y}{T_y} - l\right).$$

Nyquist: $\omega_{\text{sampling}} \geq 2\omega^{\max}$

Sampling: $f(x, y)$: $f(kT_x, lT_y)$

In the frequency domain, sampling amounts to periodization because (*Poisson* summation formula)

$$\sum_{k,l} f(kT_x, lT_y)\delta\left(\frac{x}{T_x} - k, \frac{y}{T_y} - l\right) \xrightarrow{\text{Fourier}} \sum_{k,l} F\left(\omega_x - \frac{2k\pi}{T_x}, \omega_y - \frac{2l\pi}{T_y}\right)$$

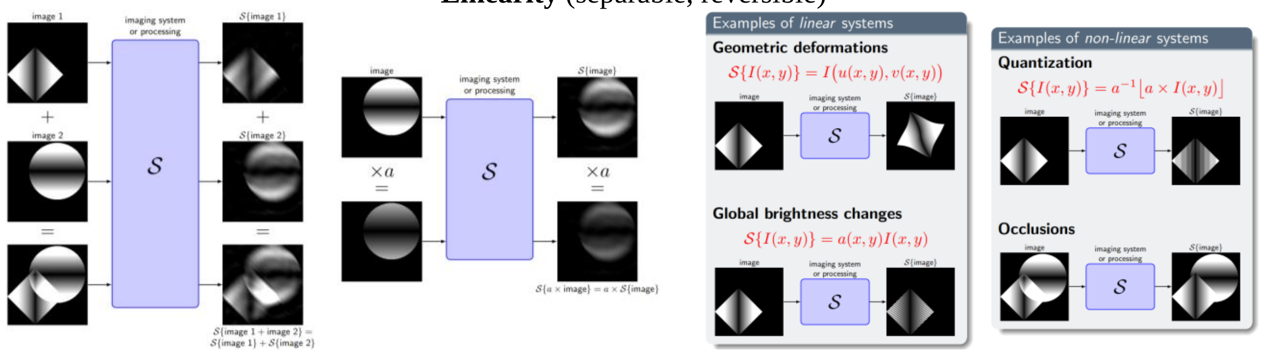


Nyquist condition is satisfied

Exact reconstruction is not possible

Invariance stuff

Linearity (separable, reversible)



Shift-invariance (not relative to an anchor point)

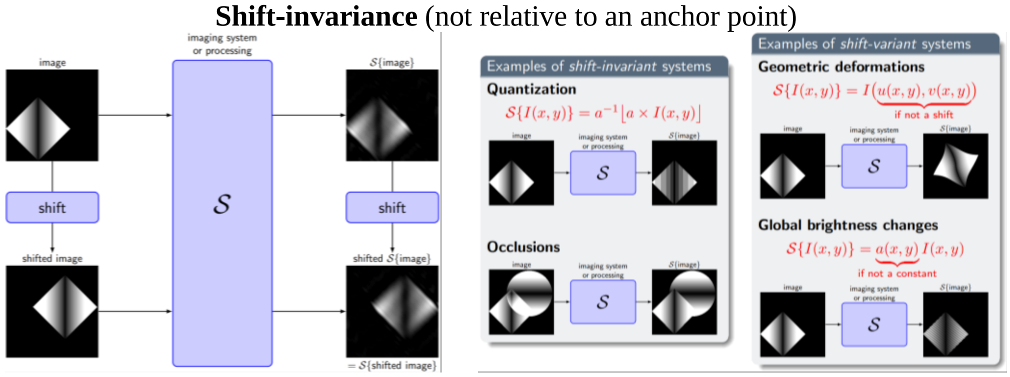
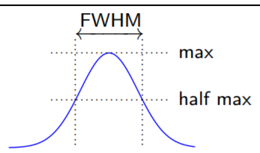


Image quality stuff

	**FWHM/Resolution** $= 2\sigma\sqrt{2\ln 2}$; $f = e^{-\frac{x^2+y^2}{2\sigma^2}}$
Modulation global contrast ratio: $m_f = \dfrac{f_{max} - f_{min}}{f_{max} + f_{min}}$	**Local contrast:** $\mathrm{C} = \left\|\dfrac{f_t - f_b}{f_b}\right\|$; $f_b = \dfrac{f_{max} + f_{min}}{2}$; $f_b = f_{max}$
Power-based SNR: $\text{SNR} = 10\log_{10}\dfrac{\text{signal variance}}{\text{noise variance}}\,\text{dB}$	**Amplitude-based SNR:** $\text{Peak SNR} = 20\log_{10}\dfrac{\text{max signal amplitude}}{\text{noise standard deviation}}\,\text{dB}$

Radiography stuff

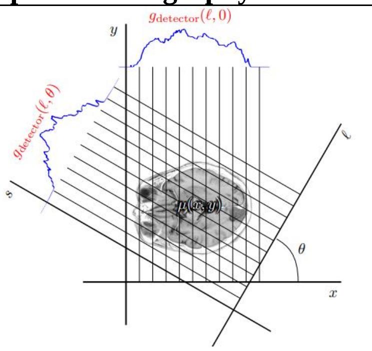
$\text{Relativistic KE} = \dfrac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$	**Beer's Law:** $I = I_0 e^{-\mu\Delta x}$	$E = hf$
Compton shift: $E' = E\left(1 + (1 - \cos\theta)\dfrac{E}{mc^2}\right)^{-1}$	$\dfrac{1}{2} = e^{-\mu\,\text{HVL}}$	$\text{HVL} = \dfrac{\ln 2}{\mu}$

H ionisation energy $= 13.6\text{eV} = 2.18 \times 10^{-18}\text{J}$	**Local contrast:** $C = \frac{I_t - I_b}{I_b}$	$\text{SNR} = C\sqrt{\text{photon count}}$

$$I(x,y) = \int_0^\infty E \cdot S_0(E) e^{-\int_0^{r(x,y)} \mu(s,E,x,y)\, ds}\, dE\, ; r(x,y) = \sqrt{d^2 + x^2 + y^2} = \frac{d}{\cos\theta}$$

1 rem = 0.87 roentgen

Computed Tomography stuff



$\uparrow: \theta = 0°$; $\nwarrow: \theta = 45°$
$\circlearrowleft : \theta \uparrow$

$$g(l,\theta) = \int \mu(l\cos\theta - s\sin\theta\, , l\sin\theta + s\sin\theta)\, ds$$

Filtered backprojection formula

Consider the 1D *"ramp" filter* $c(\ell)$ characterized by[a] $C(\omega) = |\omega|$.

Assume that what is known of the image $f(x,y)$ is its Radon projections $\mathcal{R}_\theta f$ at all angles between 0 and π radians (i.e., 0 to 180 degrees). Then, the image $f(x,y)$ can be reconstructed as follows

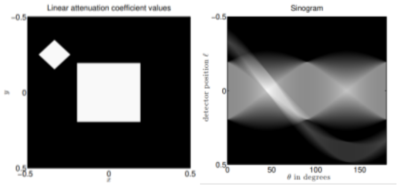
1. *Filter* the projections using the ramp filter: $h_\theta(\ell) = c(\ell) * \mathcal{R}_\theta f\,(\ell)$
2. *Backproject* the result: $f(x,y) = \frac{1}{2\pi}\int_0^\pi h_\theta(x\cos\theta + y\sin\theta)\, d\theta$

[a]calculus in the sense of distributions shows that $c(\ell) = -\frac{1}{\pi\ell^2}$.

Radon transform:

$$\mathcal{R}_\theta f(l) = \int f(l\cos\theta - s\sin\theta\, , l\sin\theta + s\sin\theta)\, ds$$
$$= \iint f(x,y)\delta(x\cos\theta + y\sin\theta - l)\, dx\, dy$$

$$\begin{cases} x = l\cos\theta - s\sin\theta \\ y = l\sin\theta + s\sin\theta \end{cases}$$



Fourier slice theorem: $\boldsymbol{\mathcal{F}}_l\big(\mathcal{R}_\theta f(l)\big) = \boldsymbol{\mathcal{F}}_{x,y}\big(f(\omega\cos\theta\, , \omega\sin\theta)\big)$

$$\mathcal{R}_\theta f(l) = \frac{1}{2\pi}\int \boldsymbol{\mathcal{F}}_{x,y}\big(f(\omega\cos\theta\, , \omega\sin\theta)\big)\, e^{j\omega l}\, d\omega$$

Hounsfield unit: $h = 1000 \times \frac{\mu - \mu_{\text{water}}}{\mu_{\text{water}}}$

$$\mathcal{R}_\theta\big(I_1 *_{x,y} I_2\big)(l) = (\mathcal{R}_\theta I_1 *_l \mathcal{R}_\theta I_2)(\theta, l)$$

$$\boldsymbol{\mathcal{F}}_l\left(\mathcal{R}_\theta\big(I_1 *_{x,y} I_2\big)(l)\right) = \boldsymbol{\mathcal{F}}_{x,y}\left(\big(I_1 *_{x,y} I_2\big)(\omega\cos\theta\, , \omega\sin\theta)\right) = \boldsymbol{\mathcal{F}}_l\big(\mathcal{R}_\theta I_1(l)\big)\boldsymbol{\mathcal{F}}_l\big(\mathcal{R}_\theta I_2(l)\big) = \boldsymbol{\mathcal{F}}_l\big((\mathcal{R}_\theta I_1 *_l \mathcal{R}_\theta I_2)(\theta, l)\big)$$

Magnetic Resonance Imaging stuff

$x \Rightarrow \text{rad}/s$; $\bar{x} \Rightarrow Hz$; $x = 2\pi\bar{x}$	$\mu = \gamma I$	$\omega = \gamma B$

Rotating frame

T_1 excitation: $M_z(t) = M_0\cos(\gamma B_1 t)$

T_2 excitation: $M_{xy}(t) = jM_0\cos(\gamma B_1 t)$

T_1 relaxation: $M_z(t) = M_0 - \big(M_0 - M_{z,initial}\big)e^{-\frac{t}{T_1}}$

T_2 relaxation: $M_{xy}(t) = M_{xy,initial}e^{-\frac{t}{T_2}}$

Laboratory frame

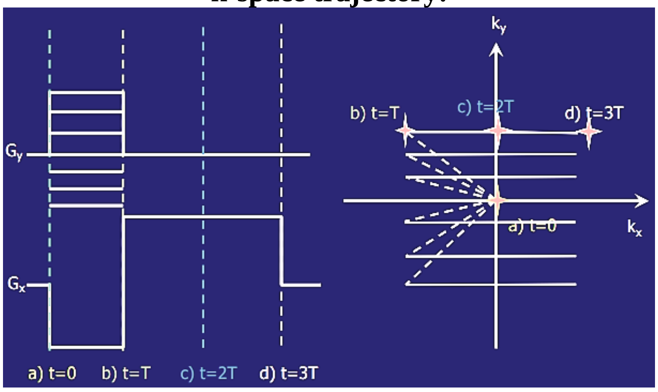
T_1 excitation: $M_z(t) = M_0\cos(\gamma B_1 t)$

T_2 excitation: $M_{xy}(t) = jM_0\cos(\gamma B_1 t)\, e^{-j\gamma B_0 t}$

T_1 relaxation: $M_z(t) = M_0 - \big(M_0 - M_{z,initial}\big)e^{-\frac{t}{T_1}}$

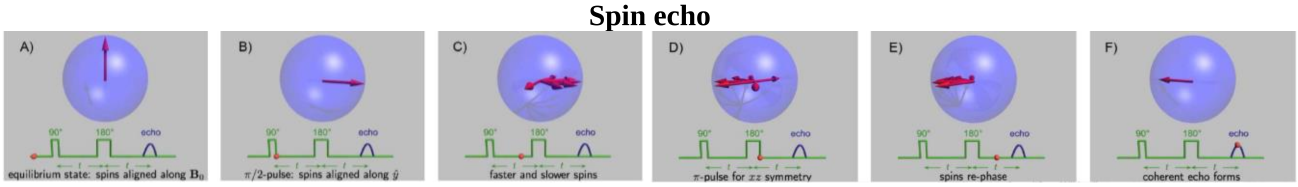
T_2 relaxation: $M_{xy}(t) = M_{xy,initial}e^{-\frac{t}{T_2}}e^{-j\gamma B_0 t}$

k-space trajectory:



G = velocity of k

Spin echo



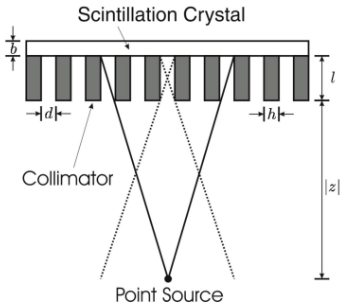
Angle linear interpolation

Larmor frequency: $\omega = \omega_0 + \gamma G_z z$; $z = \frac{\omega - \omega_0}{\gamma G_z}$	**Bandwidth:** $\Delta\omega = \gamma G_z \Delta z \Rightarrow \omega' \in \left[\omega \pm \frac{1}{2}\Delta\omega\right]$

$\Delta k = \frac{1}{\text{FOV}}$	$\Delta\omega = \frac{1}{k_{\text{FOV}}}$	$\text{FOV}_x = \frac{2\pi}{\gamma T G_x}$	$\text{FOV}_y = \frac{2\pi}{\gamma T_p \Delta G_y}$

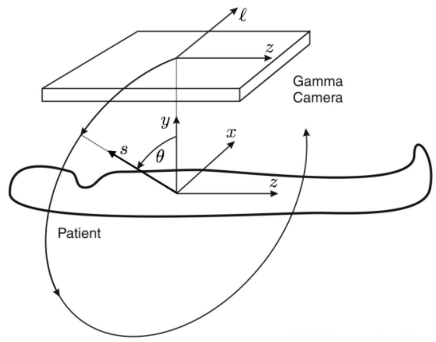
Nuclear Imaging stuff

$N(t) = N_0 e^{-\lambda t}$	$A(t) = \lambda N_0 e^{-\lambda t}$	$t_{1/2} = \frac{\ln 2}{\lambda}$	☺
$\phi(x,y) = \frac{1}{4\pi}\int_{-\infty}^{0} \frac{A(x,y,z)}{z^2} e^{-\int_z^0 \mu(x,y,z',E)\,dz'}\,dz$		$A(x,y,z) = A_{z_0}(x,y)\delta(z-z_0)$	
$Z = \sum a_k$	$(X,Y) = \frac{1}{Z}\sum a_k(x_k,y_k)$		
Resolution: $R_C(\lvert z\rvert) = \frac{d}{l}(l+b+\lvert z\rvert)$			
Gaussian Blurring PSF: $h_C(x,y,\lvert z\rvert) = e^{-4\ln 2\frac{x^2+y^2}{R_C{}^2}}$			
$\phi(x,y) = \frac{A_{z_0}(x,y)}{4\pi {z_0}^2} e^{-\int_{z_0}^{0}\mu(x,y,z_0,E)\,dz} * h_C(x,y,\lvert z_0\rvert)$			
Sensitivity: $\varepsilon = \frac{d^4}{16l^2(d+h)^2}$			
$\mathrm{SNR}_{\mathrm{intrinsic}} = \sqrt{\frac{\text{number of detected photons}}{\text{number of pixels}}}$		$\mathrm{SNR} = \text{contrast} \times \sqrt{\text{background photon fluence}}$	



SPECT stuff

$\begin{cases} x = l\cos\theta - s\sin\theta \\ y = l\sin\theta + s\cos\theta \end{cases}$
$\phi(l,\theta,z) = \frac{1}{4\pi}\int_{-\infty}^{R}\frac{A(x(s),y(s),z)}{(s-R)^2} e^{-\int_s^R \mu(x(s'),y(s'),z)\,ds'}\,ds$
Bold approxiamtion reconstruction ($\mu = 0$): ① $\phi(l,\theta,z) = \int_{-\infty}^{\infty} A(x(s),y(s),z)\,ds$ $= \iint A(x,y,z)\delta(x\cos\theta + y\sin\theta - l)\,dx\,dy$ ② Ramp filtering: $h_\theta(l,z) = c(l) *_l \phi(l,\theta,z)$ ③ Backprojection: $A(x,y,z) = \frac{1}{2\pi}\int_0^\pi h_\theta(x\cos\theta + y\sin\theta, z)\,d\theta$



PET stuff

Attenuation: $e^{-\int_{-R}^{R}\mu(x(s),y(s),z)\,ds}$	**Radioactivity:** $\int_{-R}^{R} A(x(s),y(s),z)\,ds$
$\phi(l,\theta,z) = C \times e^{-\int_{-R}^{R}\mu(x(s),y(s),z)\,ds} \times \int_{-R}^{R} A(x(s),y(s),z)\,ds$	
Attenuation correction: $\phi_C(l,\theta,z) = \phi(l,\theta,z) \times e^{\int_{-R}^{R}\mu(x(s),y(s),z)\,ds}$ $= \int_{-R}^{R} A(x(s),y(s),z)\,ds = \iint A(x,y,z)\delta(x\cos\theta + y\sin\theta - l)\,dx\,dy$	

Trigonometric identities

$\sin^2 x + \cos^2 x = 1$	$\sec^2 x - \tan^2 x = 1$	$\csc^2 x - \cot^2 x = 1$	$\frac{\sin kx}{x} = k\,\mathrm{sinc}\frac{k}{\pi}x$; $\mathrm{sinc}\,x = \frac{\sin \pi x}{\pi x}$
$\sin(x \pm y) = \sin x\cos y \pm \cos x \sin y$		$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	
$\sin 2x = 2\sin x\cos x$		$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$	
$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$	$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	$\tan x = \frac{\sin 2x}{1+\cos 2x} = \frac{1-\cos 2x}{\sin 2x} = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$	
$\sin 3x = 3\sin x - 4\sin^3 x$		$\cos 3x = 4\cos^3 x - 3\cos x$	
$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$ $\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$ $\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$		$\sin x \pm \sin y = 2\sin\frac{x\pm y}{2}\cos\frac{x\mp y}{2}$ $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$ $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$	